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Phase structure of the two-dimensional ϕ^4 field theory within thermofield dynamics

G V Efimov and S N Nedelko

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Head Post Office,
PO Box 79, SU-101000 Moscow, Russia

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Abstract. Phase structure of the two-dimensional $g\phi^4$ field theory is investigated at arbitrary coupling constant and temperature. The critical values of the coupling constant and temperature, corresponding to symmetry rearrangement in the system, are calculated by the method of canonical transformations within the formalism of thermofield dynamics. The Hamiltonians describing the system in each phase are obtained straightforwardly.

1. Introduction

In this paper we will consider the phase structure of the models

$$L(x, t) = \frac{1}{2} \phi(x, t) (\square - m^2) \phi(x, t) - \frac{1}{4} g \phi^4(x, t) \quad (1.1)$$

$$L(x, t) = \frac{1}{2} \phi(x, t) (\square + \frac{1}{2} m^2) \phi(x, t) - \frac{1}{4} g \phi^4(x, t) \quad (1.2)$$

in the two-dimensional spacetime at finite temperature.

These Lagrangians describe a one-component scalar field $\phi(x, t)$. The parameters m and g are positive. The Lagrangians are invariant under the transformation $\phi \rightarrow -\phi$. The models are super-renormalizable. The renormalization can be performed by normal ordering in the Hamiltonians.

If the dimensionless coupling constant G and the temperature θ

$$G = \frac{g}{2\pi m^2} \quad \theta = \frac{T}{m}$$

is small enough, the Lagrangian (1.1) describes the interaction symmetric under transformation $\phi \rightarrow -\phi$, but the Lagrangian (1.2) describes the situation of spontaneous symmetry breaking.

The models (1.1), (1.2) are the popular objects for investigation of dynamical symmetry reconstruction. This phenomenon is present in many profound four-dimensional theories [1].

The exact theorems strictly establish the existence of the phase transition in the model (1.1) for zero temperature [2, 3] and give the arguments that the transition should be of the second order [2-4]. But these arguments do not exclude completely the possibility of the first-order transition [3]. At the same time, these theorems do not give information about the critical value of the coupling constant G .

Fruitful attempts to construct phases in the explicit form and to calculate the approximate critical value of G were made within the Gaussian effective potential (GEP) [5-7] and beyond the GEP approximation [8].

The symmetry restoration in the model (2) at high temperature ($\theta \gg G$) was investigated within the one-loop or two-loop effective potential calculations [9, 10].

Unfortunately, variational approaches such as GEP do not allow control of the exactness of approximation [11]. The results of [9, 10] are valid only for the high temperature limit. Thus it seems interesting to construct the phases in explicit form for arbitrary θ , G and to determine for this purpose a method permitting a simple accuracy check.

We will solve the problem by the method of canonical transformations. This approach has been used for investigation of various scalar field models at zero temperature [12]. Essentially, it is a field-theoretical version of the Bogoluibov transformation method.

The apparatus of thermofield dynamics (TFD) [13, 14] provides a natural way to take into account the thermal effects within the canonical quantization approach. Since the usual operator formalism of quantum field theory can be straightforwardly extended to TFD, the method of canonical transformations keeps essentially the same structure as in the zero-temperature formulation [12].

We consider canonically quantized within TFD theories (1.1), (1.2) and formulate the problem as follows:

What representation of canonical commutation relations (CCR) is suitable for different values of the parameters G , θ and what physical picture corresponds to this representation?

Within this formulation the notions 'a phase' and 'a representation of CCR' have the same sense. According to this we define the phase transition as a transition from one representation to another.

Our approach consists in the following steps:

Step 1. We construct canonically quantized theory in representation having a suitable physical interpretation for $G \ll 1$, $\theta \ll 1$.

Step 2. We perform canonical transformations of field variables going to field with new mass and vacuum condensation and obtain a new representation of CCR.

The canonical transformation should be introduced in such a way that the total Hamiltonian in any representation (r) has the 'correct' form. This means that

$$H = H_0^{(r)} + H_1^{(r)} + R^{(r)} + VE^{(r)}.$$

Here $H_0^{(r)}$ is the standard free Hamiltonian. The interaction Hamiltonian $H_1^{(r)}$ contains the field operators in a degree more than two. The counter-terms operator $R^{(r)}$ is defined by $H_0^{(r)}$ and $H_1^{(r)}$, and it leads to cancellation of all divergencies in perturbation theory. The constant $E^{(r)}$ has a sense of a ground state energy density (here V is a large finite volume). The energy $E^{(r)}$ is connected with the free energy density like

$$F^{(r)} = E^{(r)} - \frac{T}{V} S^{(r)}$$

where $S^{(r)}$ is the the entropy of the system.

Step 3. We perform classification of CCR and choose the phase suitable for given values of G , θ . There are two mutually additional principles for this choice. A representation (r) is suitable if:

- (a) the free energy density $F^{(r)}$ is smaller in this representation, than in other ones (this is the conventional criterion);
- (b) a coupling (we may call it 'an effective coupling') is weaker in this representation than in other possible representations.

The demands of the correct form of the total Hamiltonian and the criterion of the weak effective coupling relate to the conventional scattering picture of QFT. The Hamiltonian $H_0^{(r)}$ describes the non-interacting asymptotical fields. The Hamiltonian $H_1^{(r)}$ describes the scattering of the particles and it should not contain quadratic and linear terms because they do not lead to any non-trivial scattering but only redefine the parameters of the free Hamiltonian. Conventional perturbation expansion for scattering amplitudes becomes reasonable only if the effective (perturbation) coupling constant is small enough. Thus we consider the representation as suitable if the total Hamiltonian has the correct form and the effective coupling is weak.

For two-dimensional models (1.1), (1.2) an effective coupling in any representation is given by the dimensionless coupling constant

$$G_{\text{eff}}(G, \theta) = \frac{g}{2\pi M(G, \theta)} \quad (1.3)$$

where $M(G, \theta)$ is the mass of the field in the free Hamiltonian $H_0^{(r)}$.

Incidentally, our calculations here and in [12] show that the criteria (a) and (b) do not contradict each other.

For the model (1.1) we have found that for any value of θ there is a phase transition of the first order at $G = G_c(\theta)$ accompanied by symmetry breaking (figure 1).

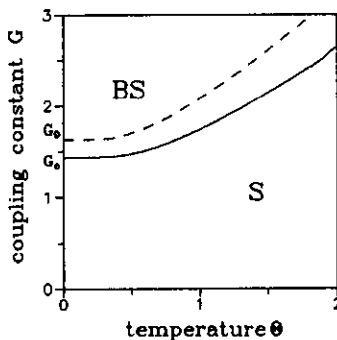


Figure 1. Phase diagram for the model (1.1). The solid line denotes $G_c(\theta)$; the dashed line is obtained by comparing the free energies.

There are two phase transitions of the first order in the system (1.2) (figure 7). One of the phase boundaries lies in the domain of applicability of high temperature expansion ($\theta \gg G$) and it is in good quantitative agreement with existing results [9, 10, 16]. The symmetry is restored in both systems (1.1) and (1.2) for any fixed value of G , if the temperature is high enough.

According to (a) we find the phase boundaries as the points of equality of the free energy density for the S- and BS-phases. The critical values of G at zero temperature, obtained in this way, coincide with the numerical results of the GEP-approximation [5, 7].

For arbitrary θ the regime of strong coupling is absent in both models (1.1), (1.2). The effective coupling constant $G_{\text{eff}}(G, \theta)$ is small for any G, θ excluding the critical regions where it is $o(1)$. The function $G_{\text{eff}}(G, \theta)$ decreases if G increases.

In other words our approximation is good enough and hence we have an accurate quantum field-theoretical description of the systems, unless the values of G, θ are

outside the critical regions. The method we use becomes rough in these regions and our result that all transitions are of the first order cannot be considered as well established. Thus we can neither confirm nor contradict that the transitions are of the second order.

Our calculations show that the renormalization structure of the models plays a crucial role in forming their phase structure both at zero and finite temperature. It agrees with the following heuristic motivation (see the monograph by Simon in [2]). Intuitively clear reason for symmetry breaking in $(\phi^4)_2$ comes from the normal operator ordering, in other words from the mass renormalization arising from the tadpole diagram. Normal ordering makes the potential of the model a two-well type. As a consequence the system can transit from the symmetric phase to the non-symmetric one, if the coupling constant is large enough.

This motivation is valid both for zero and finite temperature, since the thermal effects do not change the renormalization structure of the models.

2. Thermofield dynamics

It is convenient to deal with a general form of the Lagrangian density

$$L(x, t) = \frac{1}{2} \phi(x, t) (\square - m^2) \phi(x, t) - g' \phi^3(x, t) - \frac{1}{4} g \phi^4(x, t). \tag{2.1}$$

Putting $g' = 0$ we get the Lagrangian (1.1), but if $g' = m\sqrt{g/2}$ then we get the Lagrangian (1.2), shifted to the minimum of the classical potential (a constant term we will omit).

The detailed description of TFD can be found, for example, in [13]. We confine ourselves to the brief formulation.

The total Hamiltonian for the system (2.1) has the following form in the TFD-formalism: $\hat{H} = H - \tilde{H}$ where

$$H = H_0 + H_1 \tag{2.2}$$

$$H_0 = \frac{1}{2} \int dx: [\pi^2(x) + (\nabla \phi(x))^2 + m^2 \phi^2(x)]:$$

$$H_1 = \int dx: \left[\frac{g}{4} \phi^4(x) + g' \phi^3(x) \right]:$$

$$\phi(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega(k)}} [a(k) e^{ikx} + a^+(k) e^{-ikx}] \tag{2.3}$$

$$\pi(x) = i \int \frac{dk}{\sqrt{4\pi}} \sqrt{\omega(k)} [a(k) e^{ikx} - a^+(k) e^{-ikx}] \tag{2.4}$$

$$\tilde{H} = H^*[\tilde{\phi}, \tilde{\pi}]$$

$$\tilde{\phi}(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega(k)}} [\tilde{a}(k) e^{ikx} + \tilde{a}^+(k) e^{-ikx}] \tag{2.5}$$

$$\tilde{\pi}(x) = \frac{1}{i} \int \frac{dk}{\sqrt{4\pi}} \sqrt{\omega(k)} [\tilde{a}(k) e^{ikx} - \tilde{a}^+(k) e^{-ikx}] \tag{2.6}$$

$$\omega(k) = \sqrt{k^2 + m^2}$$

$$[a(k), a^+(k')] = \delta(k - k') \quad [\tilde{a}(k), \tilde{a}^+(k')] = \delta(k - k'). \tag{2.7}$$

Here the Hamiltonian \hat{H} is defined on the Fock space with vacuum vector given by

$$a(k)|o\rangle \otimes |o\rangle = \tilde{a}(k)|o\rangle \otimes |o\rangle = 0$$

where $|o\rangle$ is the ground state of the field system at zero temperature.

The temperature-dependent operators $\alpha(k, \beta)$, $\tilde{\alpha}(k, \beta)$ are introduced by the canonical transformation:

$$a(k, \beta) = a(k) \cosh(\chi(k, \beta)) - \tilde{a}^+(k) \sinh(\chi(k, \beta))$$

$$\tilde{a}(k, \beta) = \tilde{a}(k) \cosh(\chi(k, \beta)) - a^+(k) \sinh(\chi(k, \beta))$$

with the inverse transformation given by

$$a(k) = \alpha(k, \beta) \cosh(\chi(k, \beta)) + \tilde{\alpha}^+(k, \beta) \sinh(\chi(k, \beta)) \quad (2.8)$$

$$\tilde{a}(k) = \tilde{\alpha}(k, \beta) \cosh(\chi(k, \beta)) + \alpha^+(k, \beta) \sinh(\chi(k, \beta)). \quad (2.9)$$

This is the Bogoliubov transformation and it can be performed in the operator form

$$\alpha(k, \beta) = U^{-1}(\beta)a(k)U(\beta) \quad \tilde{\alpha}(k, \beta) = U^{-1}(\beta)\tilde{a}(k)U(\beta)$$

where

$$U(\beta) = \exp\left\{ \int dk \chi(k, \beta) [\tilde{a}(k)a(k) - a^+(k)\tilde{a}^+(k)] \right\}. \quad (2.10)$$

Although the operators such as (2.10) are only defined in finite volume, they are so useful (for example, in determining the forms of Bogoliubov transformations) that we will freely use them in the following discussion with the implicit understanding of the space cutoff. It is only important for us that such operators determine the canonical transformations.

A ground state of the field system at temperature $T = 1/\beta$ is defined by the relations

$$\alpha(k, \beta)|o(\beta)\rangle = 0$$

$$\tilde{\alpha}(k, \beta)|o(\beta)\rangle = 0.$$

The parameter $\chi(k, \beta)$ is defined by

$$\sinh(\chi(k, \beta)) = [\exp(\beta\omega) - 1]^{-1/2}. \quad (2.11)$$

The connection between TFD and zero-temperature field theory is given essentially by the relations

$$|o(\beta)\rangle = U^{-1}(\beta)|o\rangle \otimes |o\rangle$$

$$a(k) = \bar{a}(k) \otimes \mathbb{1} \quad \tilde{a}(k) = \mathbb{1} \otimes \bar{a}(k)$$

$$H = \bar{H} \otimes \mathbb{1} \quad \tilde{H} = \mathbb{1} \otimes \bar{H}.$$

They reflect the necessity to use the quasiparticle picture for the description of a quantum field system at finite temperature. The state $|o\rangle$ and the operators $\bar{a}(k)$, \bar{H} , are the ground state, the annihilation operator and the Hamiltonian at zero temperature.

Due to (2.11) the vacuum expectation value of any observable coincides with its statistical average. In particular, the expectation value $\langle o(\beta)|H_o|o(\beta)\rangle$ should be equal to the energy of the ideal Bose gas in equilibrium. Such normalization relates to the normal ordering of the operators $a^+(k)$, $a(k)$ in the Hamiltonian (2.2) (rather than $\alpha^+(k, \beta)$, $\alpha(k, \beta)$).

3. Canonical transformation

Here we proceed in the following way:

Step 1. We construct a new unitary non-equivalent representation of the relations (2.7) by a canonical transformation of the operators $a(k)$ and $\tilde{a}(k)$.

Step 2. We fix the parameters of the canonical transformation by a requirement providing the correct form of a new Hamiltonian.

Step 3. We choose a representation, which is suitable for given G, θ according to our criteria (a), (b).

Step 4. We calculate perturbation corrections to the free energy density using the Hamiltonians in different representations and reform the phase picture.

As the first step we perform the canonical transformation

$$a(k) \rightarrow a_f(k) - \sqrt{2\pi} m B \delta(k) = U_2^{-1}(f) U_1^{-1}(B) a(k) U_1(B) U_2(f)$$

where

$$U_1(B) = \exp\left\{-\sqrt{2\pi} m B \int dk \delta(k) [a(k) - a^+(k)]\right\}$$

$$U_2(f) = \exp\left\{\frac{1}{2} \int dk \lambda(k, f) [a(-k)a(k) - a^+(k)a^+(-k)]\right\}$$

the quantity B is a constant.

The operator U_2 has the same structure as $U(\beta)$ (2.10). It is easy to obtain the following relations:

$$a(k) = a_f(k) \cosh(\lambda(k, f)) + a_f^+(-k) \sinh(\lambda(k, f))$$

$$a^+(k) = a_f^+(k) \cosh(\lambda(k, f)) + a_f(-k) \sinh(\lambda(k, f))$$
(3.1)

which are analogous to (2.8), (2.9).

If the parameter λ has the form

$$\lambda(k, f) = \frac{1}{2} \ln \left[\frac{\omega(k)}{\omega_f(k)} \right]$$

$$\omega_f(k) = \sqrt{k^2 + M^2} \quad M^2 = m^2(1 + f)$$

then using the relations (3.1) one can obtain a new representation for the fields $\phi(x)$ and $\pi(x)$ ((2.3), (2.4))

$$\phi(x) = \int \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega_f}} [a_f(k) e^{ikx} + a_f^+(k) e^{-ikx}] \equiv \phi_f(x)$$
(3.2)

$$\pi(x) = i \int \frac{dk}{\sqrt{4\pi}} \sqrt{\omega_f} [a_f(k) e^{ikx} - a_f^+(k) e^{-ikx}] \equiv \pi_f(x).$$

The U_1 -transformation leads to the shifting of the field ϕ at the constant B , thus

$$\phi = \phi_f + B.$$
(3.3)

Analogous transformation are performed for $\tilde{a}(k)$ in order to obtain $\tilde{a}_f(k)$.

The operators a_f, \tilde{a}_f annihilate the state

$$|0\rangle = U_2^{-1}(f) \tilde{U}_2^{-1}(f) U_1^{-1}(B) \tilde{U}_1^{-1}(B) |0\rangle \otimes |0\rangle$$

where \bar{U} -transformations coincide with $U_{1(2)}$ up to the substitution of \tilde{a}, \tilde{a}^+ for a, a^+ . This equation is a formal relation between the ground states of two unitary non-equivalent representations.

In order to consider the thermal effects we follow the above-mentioned TFD-prescriptions to obtain the relation

$$a_f(k) = \alpha_f(k, \beta) \cosh(\chi_f(k, \beta)) + \tilde{\alpha}_f^+(k, \beta) \sinh(\chi_f(k, \beta)) \quad (3.4)$$

where $\sinh(\chi_f(k, \beta)) = [\exp(\beta\omega_f) - 1]^{-1/2}$.

The operators $\alpha_f, \tilde{\alpha}_f$ annihilate the state

$$|0(\beta)\rangle\rangle = U_f^{-1}(\beta)|0\rangle\rangle$$

where

$$U_f(\beta) = \exp\left\{\int dk \chi_f(k, \beta) [\tilde{a}_f(k) a_f(k) - a_f^+(k) \tilde{a}_f^+(k)]\right\}.$$

Using equations (3.2) and (3.3) we now express the Hamiltonian (2.2) in terms of ϕ_f, π_f and then taking into account (3.4) go to the normal ordering of the operators $\alpha_f, \tilde{\alpha}_f$. As a result we obtain the expression

$$H = VE + H'_0 + H'_1 + H_1$$

where

$$H'_0 = \frac{1}{2} \int dx: [\pi_f^2(x) + (\nabla\phi_f(x))^2 + M^2\phi_f^2(x)]:$$

$$H'_1 = \int dx: \left[\frac{g}{4} \phi_f^4(x) + (gB + g')\phi_f^3(x) \right]: \quad (3.5)$$

$$H_1 = \int dx: \left[\frac{1}{2} \phi_f^2(x) R(f, B) + \phi_f(x) P(f, B) \right]:$$

$$R(f, B) = -m^2 f + 3gB^2 - 3gD_\theta(f) + 3g'B$$

$$P(f, B) = Bm^2 + gB[B^2 - 3D_\theta(f)] + 3g'[B^2 - D_\theta(f)] \quad (3.6)$$

$$E = \frac{1}{2} m^2 B^2 + L_\theta(f) + \frac{g}{4} [B^4 - 6B^2 D_\theta(f) + 3D_\theta^2(f)] + g' B [B^2 - 3D_\theta(f)] \quad (3.7)$$

$$L_\theta(f) = \frac{m^2}{8\pi} \{f - 4\pi D_\theta(f) + 4(1+f)[2s(\theta/\sqrt{1+f}) + d(\theta/\sqrt{1+f})]\}$$

$$D_\theta(f) = \frac{1}{4\pi} \ln(1+f) - \frac{1}{\pi} d(\theta/\sqrt{1+f})$$

$$d(z) = \int_0^\infty \frac{du}{\sqrt{1+u^2}} \left(\exp\left\{\frac{1}{z}\sqrt{1+u^2}\right\} - 1 \right)^{-1}$$

$$s(z) = \int_0^\infty \frac{du u^2}{\sqrt{1+u^2}} \left(\exp\left\{\frac{1}{z}\sqrt{1+u^2}\right\} - 1 \right)^{-1}$$

$$d(0) = 0 \quad s(0) = 0.$$

Since the symbol of normal ordering in (3.5) relates to the operators $\alpha_f, \tilde{\alpha}_f$, it is obvious that the quantity E given by (3.7) is an estimation from above on the energy density of the state $|\alpha(\beta)\rangle\rangle$. The free energy density is given by

$$F = E - \frac{1}{V\beta} S \quad (3.8)$$

where the entropy S has the form

$$S = -V \int \frac{dk}{\sqrt{2\pi}} [n_f(k) \cdot \ln(n_f(k)) - (1 - n_f(k)) \ln(1 - n_f(k))] \\ n_f(k) = [\exp(\beta\omega_f) - 1]^{-1}.$$

The expression for S can be rewritten as

$$S = \frac{m^2}{\pi} V\beta(1+f)[2s(\theta/\sqrt{1+f}) + d(\theta/\sqrt{1+f})]. \quad (3.9)$$

Using equations (3.7), (3.8) and (3.9) we obtain the expression for the free energy density

$$F = \frac{1}{2}m^2B^2 + \frac{g}{4}[B^4 - 6B^2D_\theta(f) + 3D_\theta^2(f)] + g'B[B^2 - 3D_\theta(f)] \\ + \frac{m^2}{8\pi}\{f - 4\pi D_\theta(f) - 4(1+f)[2s(\theta/\sqrt{1+f}) + d(\theta/\sqrt{1+f})]\}. \quad (3.10)$$

Let us put the coefficients R, P equal to zero

$$R(f, B) = 0 \quad P(f, B) = 0. \quad (3.11)$$

This requirement leads to $H_1 = 0$ and, hence, provides the correct form of the Hamiltonian (see introduction).

The function $D_\theta(f)$ appeared in (3.6), (3.10) as a result of normal operator ordering. It is given by

$$D_\theta(f) = \Delta(0; m^2) - \Delta_\theta(0; m^2(1+f))$$

where $\Delta(x, t; m^2)$ and $\Delta_\theta(x, t; m^2(1+f))$ are zero temperature propagator with the mass m^2 and temperature dependent propagator with the mass $m^2(1+f)$ respectively. In other words, $D_\theta(f)$ arises from the mass renormalization and contributes to equations (3.10), (3.11) describing the phases. There is the correlation between the renormalization and phase structure of the model.

It is easy to check the equivalence of (3.11) to the equations

$$\frac{\partial F(f, B)}{\partial B} = 0 \\ \frac{\partial^2 F(f, B)}{\partial B^2} = M^2 = m^2(1+f)$$

which are analogous to the minimum and stability conditions for the effective potential [1].

We stress that equations (3.11) define the minimum of the free energy (3.10) as a function of two variables f, B and for $\theta = 0$. Hence, only at $\theta = 0$ our and GEP numerical results [5, 7] should coincide. At finite temperature our technique differs from the variational approach.

4. The symmetric model

Putting the constant g' equal to zero and using equations (3.11) and (3.6) the following equations for f and B are obtained:

$$\begin{aligned} B[gB - 3gD_\theta(f) + m^2] &= 0 \\ 3gB^2 - 3gD_\theta(f) - m^2f &= 0. \end{aligned} \quad (4.1)$$

There are two phases in correspondence to two solutions of the system (4.1).

4.1. The symmetric under $\phi_f \rightarrow -\phi_f$ phase (S)

Putting $B = 0$ in the second equation (4.1) we obtain

$$\frac{2}{3G}f = -\ln(1+f) + 4d(\theta/\sqrt{1+f}). \quad (4.2)$$

This equation defines the mass M of the field and has a unique solution for any G, θ . Using the relations (3.5), (3.10) and (4.2) we obtain the Hamiltonian and free energy for this phase

$$\begin{aligned} H'_0 &= \frac{1}{2} \int dx: [\pi_f^2(x) + (\nabla\phi_f(x))^2 + M^2\phi_f^2(x)]; \\ H'_1 &= \int dx: \left[\frac{g}{4}\phi_f^4(x) \right]; \\ F_s &= \frac{m^2}{8\pi} \left\{ \left[\frac{2}{3G} + 1 \right] f + \frac{f^2}{3G} - 4(1+f)[2s(\theta/\sqrt{1+f}) + d(\theta/\sqrt{1+f})] \right\}. \end{aligned} \quad (4.3)$$

4.2. The phase with broken symmetry

Using the non-zero solution for B we can rewrite (4.1) in the form

$$\begin{aligned} B^2 &= \frac{1+f}{4\pi G} \\ \frac{1}{3G}f + \frac{1}{G} &= \ln(1+f) - 4d(\theta/\sqrt{1+f}). \end{aligned} \quad (4.4)$$

The second equation in (4.4) defines the mass of the field. A numerical analysis shows that this equation has a real solution only for θ, G such that $G \geq G_c(\theta)$. The function $G_c(\theta)$ is given in figure 1 by the solid line. Using equations (3.5), (3.10) and (4.4) we obtain the Hamiltonian and the free energy in the BS -phase

$$\begin{aligned} H'_1 &= \int dx: \left[\frac{g}{4}\phi_f^4(x) \pm gB\phi_f^3 \right]; \\ F_B &= \frac{m^2}{8\pi} \left\{ -\frac{1}{2G} + \left[1 - \frac{1}{3G} \right] f - \frac{f^2}{6G} - 4(1+f)[2s(\theta/\sqrt{1+f}) + d(\theta/\sqrt{1+f})] \right\}. \end{aligned} \quad (4.5)$$

The free Hamiltonian has the standard form (4.3). The value of condensate B plays the role of a parameter of order.

At the last step of our consideration we choose a phase which is suitable for a given θ, G .

Comparing the effective coupling constants $G_{\text{eff}}(G, \theta)$ (see (1.3)) in the *S*- and *BS*-phases we find that the phase-stable boundary is given by the same function $G_c(\theta)$ (solid line in figure 1). As soon as the solution for the *BS*-phase exists, the coupling in this phase is weaker than in the symmetric phase (figure 2).

A comparison of the free energy densities F_S and F_B leads to the boundary given by the dashed line in figure 1. The value $G_0 = 1.625 \dots$ coincides with the critical coupling constant in the GEP-approximation [5, 6] since at $\theta = 0$, and only in this case, equations (4.1), (4.2) coincide with the equations minimizing the Gaussian effective potential.

One can see that phase boundaries obtained by these two methods do not contradict each other.

The effective coupling constant is large enough in both phases in the critical region (see figure 2), hence, perturbation corrections have to be large and can change the boundary given by the dashed line in figure 1.

Let us calculate these corrections to the free energy densities F_S and F_B at zero temperature, for simplicity. For this purpose we use the Hamiltonians given in (4.3), (4.5). We take into account the corrections up to $o(G^3)$ for the phase *S* and up to $o(G_{\text{eff}}^2)$ for the phase *BS*. The necessary diagrams are shown in figures 3 and 4. The result turns out to be equal

$$\Delta F_S = \frac{m^2}{8\pi} (-1.671G^2 + 4.0388G^3 + o(G^4)) \tag{4.6}$$

$$\Delta F_B = \frac{M^2}{8\pi} (-1.758G_{\text{eff}} - 4.316G_{\text{eff}}^2 - o(G_{\text{eff}}^3)) \tag{4.7}$$

for *S* and *BS* respectively.

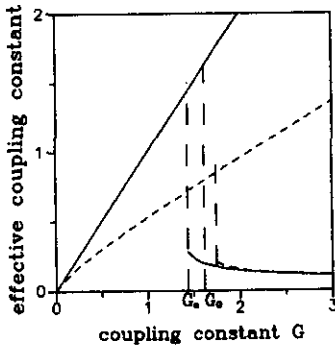


Figure 2. Effective coupling constants for the model (1.1). Upper lines relate to the *S*-phase. The case $\theta = 1$ is given by the dashed line.

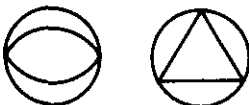


Figure 3. The diagrams $o(G^2), o(G^3)$ for *S*-phase.

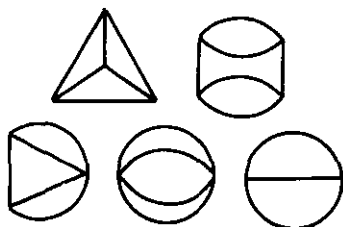


Figure 4. The diagrams $o(G_{eff}), o(G_{eff}^2)$ for BS-phase.

One can see that (4.6) represents the alternating asymptotic series. The series (4.7) is non-alternating. This is the usual situation for a system with degenerate vacuum (see for example [15]). So we can make a Borel summation of (4.6). This is not so for (4.7) because of singularity on the integration contour. Nevertheless, we make the summation defining the Cauchy mean value of the integral. The result is

$$\Delta F_S = \frac{m^2}{8\pi} \left(\int_0^\infty dt e^{-t} \frac{1 + 0.8057 Gt}{1 + 0.8057 Gt + 0.8355 (Gt)^2 - 1} \right)$$

$$\Delta F_B = -\frac{M^2}{8\pi} \left(\text{VP} \int_0^\infty dt e^{-t} \frac{1 + 0.5305 G_{eff}t}{1 - 1.2275 G_{eff}t} - 1 \right).$$

The solid and dashed lines in figure 5 represent the free energy density without and with the perturbative corrections, respectively. One can see that the corrections

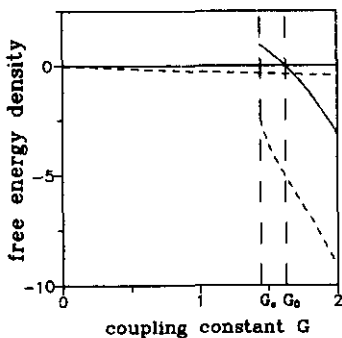


Figure 5. Free energy densities for the model (1.1). Solid lines correspond to F_S, F_B , dashed lines denote $F_S + \Delta F_S, F_B + \Delta F_B$.

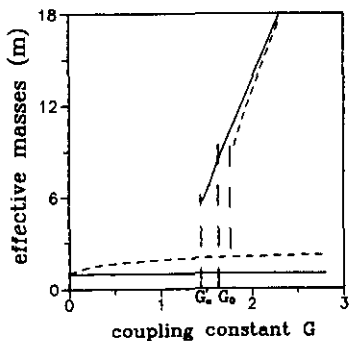


Figure 6. The masses for the model (1.1). Solid lines correspond to $\theta = 0$, dashed lines $\theta = 1$. Upper lines relate to the BS-phase.

'shift' the critical point from $G_A \approx 1.625$ to $G_c(0) \approx 1.44$. The analogous picture should take place for non-zero temperature either.

The order parameter and the mass $M(G, \theta)$ (figure 6) are discontinuous at the boundary; hence, the phase transition is of the first order. This conclusion is not well established since the coupling constant G_{eff} is large enough in the critical region.

At the same time, since $G_{\text{eff}}(G, \theta)$ is smaller than $G_c(0) = 1.44 \dots$ and decreases when G increases, we conclude that the strong coupling regime does not exist in model (1) (see figure 2) and our description of the system is accurate outside the critical region.

5. The two-well potential

Using once more equations (3.6) but with $g' = m\sqrt{g/2}$ we obtain for (3.11) the following form:

$$gB^3 + 3m\sqrt{g/2}B^2 + B[m^2 - 3gD_\theta(f)] - 3m\sqrt{g/2}D_\theta(f) = 0$$

$$3gB^2 + 3m\sqrt{2g}B - 3gD_\theta(f) - m^2f = 0. \tag{5.1}$$

According to the solutions of this system there are two phases with broken symmetry and one being symmetric.

5.1. S-phase

The first equation in (5.1) has a solution

$$B = -\frac{1}{\sqrt{4\pi G}}. \tag{5.2}$$

Using (5.2) in the second equation (5.1) we obtain the relation defining the mass of the field in the S-phase

$$\frac{2}{3G}f + \frac{1}{G} = -\ln(1+f) + 4d(\theta/\sqrt{1+f}). \tag{5.3}$$

The unique solution exists for any G, θ . Relations (3.5), (3.10) and (5.2), (5.3) lead to the Hamiltonian in the form (4.3) and to the following free energy density:

$$F_S = \frac{m^2}{8\pi} \left\{ \frac{1}{2G} + \left[\frac{2}{3G} + 1 \right] f + \frac{f^2}{3G} - 4(1+f)[2s(\theta/\sqrt{1+f}) + d(\theta/\sqrt{1+f})] \right\}.$$

5.2. BS-phases

Using the rest solutions of the first equation (5.1)

$$B = -\frac{1 \pm \sqrt{1+f}}{\sqrt{4\pi G}} \tag{5.4}$$

we obtain the equation for $M(G, \theta)$

$$\frac{1}{3G}f = \ln(1+f) - 4d(\theta/\sqrt{1+f}). \tag{5.5}$$

This equation has solutions only for θ, G such that $G \leq G_c^{(1)}(\theta)$ or $G \geq G_c^{(2)}(\theta)$. The functions $G_c^{(1)}$ and $G_c^{(2)}$ are given in figure 7 by the solid lines. There are two solutions and they are equal to each other for $G = G_c^{(1)}(\theta)$ or $G_c^{(2)}(\theta)$. These solutions represent

two different phases with broken symmetry. Let us demonstrate this for the case $\theta = 0$. Since $d(0) = 0$, one can see that (5.5) has a trivial solution $f = 0$ for any G , leading to the initial representation with the mass m and with the vacuum condensate $\sigma = \pm 1/\sqrt{4\pi G}$. This is the first BS-phase. In order to exclude this trivial solution from (5.5) we rewrite the latter in the form ($\theta = 0$)

$$\frac{1}{3G} = \frac{\ln(1+f)}{f} \tag{5.6}$$

This equation has a unique solution for any G and defines the second BS-phase with the mass and vacuum condensate given by

$$M^2 = m^2(1+f) \quad \sigma = \pm \sqrt{\frac{1+f}{4\pi G}}$$

One can see from (5.6) that

$$M^2 = m^2 \exp\left\{-\frac{1}{3G}\right\} \text{ as } G \rightarrow 0.$$

Non-analyticity of M at the point $G = 0$ means that the difference between m and M cannot be obtained within the perturbative calculations and that the first and second solutions of (5.5) represent two different phases.

One can see from figure 7 that $G_c^{(1)}(0) = G_c^{(2)}(0) = G_c$. Using $f = 0$ in (5.6) we obtain the value $G_c = \frac{1}{3}$. The region at the phase plane (figure 7) below the boundary $G_c^{(1)}(\theta)$ corresponds to the first BS-phase, but the region above the boundary $G_c^{(2)}(\theta)$ represents the second BS-phase.

The free Hamiltonian has a standard form. The interaction Hamiltonian and the free energy density in the BS-phases take the following form

$$H'_I = \int dx: \left[\frac{g}{4} \phi_f^4(x) \pm g\sigma\phi_f^3 \right];$$

$$F_B = \frac{m^2}{8\pi} \left\{ \left[1 - \frac{1}{3G} \right] f - \frac{f^2}{6G} - 4(1+f)[2s(\theta/\sqrt{1+f}) + d(\theta/\sqrt{1+f})] \right\}.$$

At the last step we choose the phase suitable for a given θ, G .

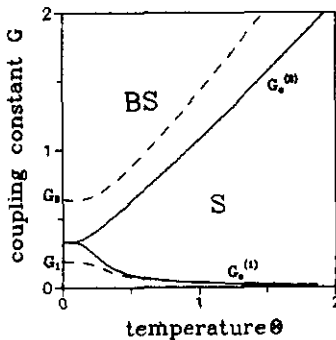


Figure 7. Phase diagram for the model (1.2). Dashed lines are obtained by comparing the free energies.

The phase-stable boundaries, obtained by comparing the free energies (dashed lines in figure 7) and the effective coupling constants (solid lines in figure 7) do not contradict each other. The values $G_1 = 0.19\dots$ and $G_2 = 0.64\dots$ in figure 7, coincide with the result of the GEP-approximation [7]; the reason has been explained in the previous section.

Let us discuss the phase picture, given by the functions $G_c^{(1)}(\theta)$ and $G_c^{(2)}(\theta)$ (solid lines in figure 7). There are two phases with broken symmetry and one which is symmetric. At zero temperature the symmetry is broken for any G , but a phase transition without symmetry rearrangement takes place at $G_c = \frac{1}{3}$. The symmetry is restored for any fixed value of G if the temperature is high enough.

The phase transitions are of the first order since the mass and the order parameter σ are discontinuous at the phase boundaries, as in figure 8 (dashed line for $\theta = 1$).

The effective coupling constant $G_{\text{eff}}(G, \theta)$ is small unless (G, θ) is outside the critical regions where G_{eff} is $\mathcal{O}(1)$ (see figure 9). This means that our description of the system is good enough everywhere, except at the locality of the phase transitions.

The phase boundary $G_c^{(1)}(\theta)$ lies in the region $\theta \gg G$ of applicability of high temperature expansion. Behaviour of this boundary is in good quantitative agreement with the results obtained by high temperature expansion [9, 10, 16].

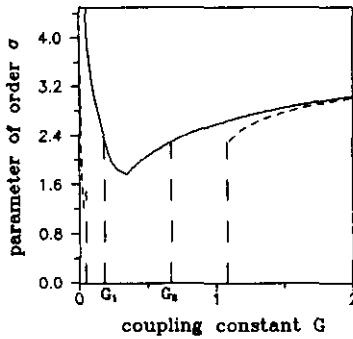


Figure 8. The order parameter for the model (1.2). The solid line relates to $\theta = 0$, dashed lines $\theta = 1$.

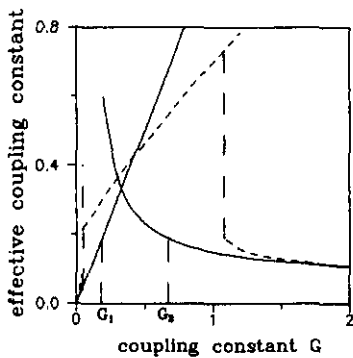


Figure 9. Effective coupling constants for the model (1.2). Solid lines correspond to $\theta = 0$, the case $\theta = 1$ is given by the dashed lines. The upper dashed line relates to the S-phase.

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